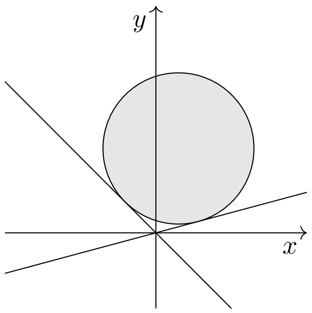
4101. A function is given, over $\mathbb{R}$, as $f(x) = \sin x + x$, where the sine function takes radian input.

(a) Show that the graph of $y = f(x)$ has infinitely many stationary points of inflection.

(b) Hence, sketch $y = f(x)$.

4102. A smooth cylinder of weight W, represented in cross-section by a circle of radius 1 in a vertical (x, y) plane, rests in equilibrium on two oblique planes, represented in cross-section by the lines $y = -x$ and $y = (2 - \sqrt{3})x$.



Find the contact force applied to the cylinder by each of the two planes.

4103. Three functions, all defined over $\mathbb{R}$, are such that

$$f_0'(x) = f_1(x),$$
$$f_1'(x) = f_2(x).$$

Find $\int x\, f_2(x)\, dx$ in terms of $f_0(x)$ and $f_1(x)$.

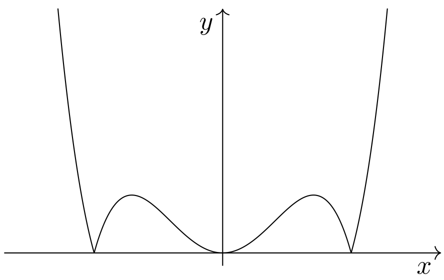
4104. For non-zero constants $a_1, ..., a_k \in \mathbb{R}$ and $k \in \mathbb{N}$, a function has the form

$$f(x) = a_1 x + a_2 x^3 + a_3 x^5 + ... + a_k x^{2k+1}.$$

Prove that $y = f(x)$ is inflected at the origin.

4105. A triangle has side lengths in GP with common ratio r. Determine all possible values of r.

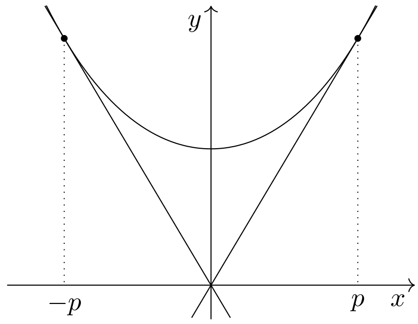
4106. The graph shown is $y = |x^4 - x^2|$.



Sketch the graph of $|y| = |x^4 - x^2|$.

4107. Find $\displaystyle\int \frac{e^x + 2}{e^x + 3}\, dx$.

4108. The tangents to the curve $y = e^x + e^{-x}$ at $x = \pm p$ pass through the origin.



(a) Show that $e^{2p} - \dfrac{2}{p-1} - 1 = 0$.

(b) Hence, find p to 3sf.

4109. Show that the curve $y = \sin^2 x$ and the x axis enclose infinitely many regions of area $\frac{\pi}{2}$.

4110. You are given that the function

$$f(x) = \ln x + \ln(k - x)$$

is stationary at $f(x) = \ln 25$. Determine k.

4111. A circle and a parabola are defined, for constants $a, b > 0$, by the equations

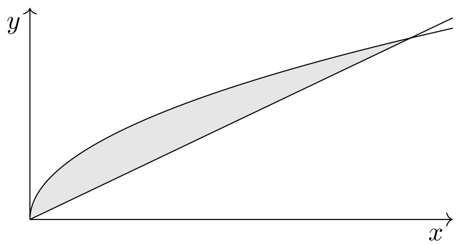
$$x^2 + y^2 = 1,$$
$$y = a - bx^2.$$

You are given that the circle and the parabola have exactly three distinct points of intersection. Find all possible values of a and b.

4112. The numbers $a, b, c, d, e \in \mathbb{R}$ are in increasing GP, with common ratio $r > 1$. Prove that

$$a + e > b + d > 2c.$$

4113. The area enclosed by the curve $y = \sqrt{x}$ and the line $y = mx$ is 36.



Determine the value of m.

4114. From a small population of a hundred, a sample of four is taken. Assuming that exactly 90% of the population is below the 90^{th} percentile, find, to 3sf, the probability that exactly two of the four are at or above the 90th percentile.

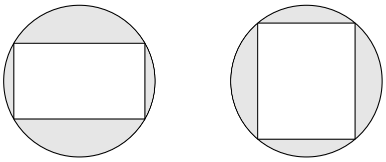
4115. A graph $y = \mathrm{h}(x)$ has a point of inflection at (a, b). For each of the following, state whether the graph must necessarily have a point of inflection. If so, give the coordinates of the point of inflection; if not, give a counterexample.

(a) $y = 2\,\mathrm{h}(x) + 3$,

(b) $y = \big(\mathrm{h}(x)\big)^2$,

(c) $y = \mathrm{h}(2x + 3)$.

4116. A rectangle is to be inscribed in a fixed circle.



Prove that the area of the rectangle is maximised if it is a square.

4117. Find $\displaystyle\int x^2 \ln x \, dx$.

4118. An implicit relationship is given as

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1.$$

Show that the locus of this relationship contains part, but not all, of the unit circle.

4119. Solve the equation $x^{\frac{3}{2}} + 3x = x^{\frac{1}{2}} + 3$.

4120. Consider the curve $y = \dfrac{1-x^2}{1+x^2}$.

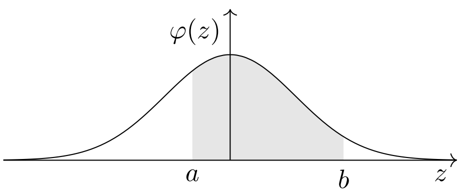
(a) Show that the curve has one stationary point.

(b) Determine the behaviour as $x \to \infty$.

(c) Hence, sketch the curve.

4121. For a normal distribution $Z \sim N(0, 1)$, computers calculate $\mathbb{P}(a < Z < b)$ by evaluating numerically the definite integral

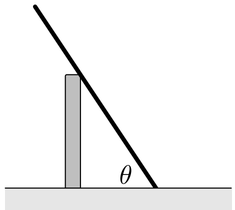
$$\mathbb{P}(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz.$$



(a) Use the trapezium rule with four strips each of width 0.25 to estimate $\mathbb{P}(0 < Z < 1)$.

(b) Compare your answer to the value produced by a calculator, with reference to the "bell curve" shape of the normal distribution.

4122. A ladder of length $2l$ and weight W is placed, in equilibrium, against a low wall of height l. The ground is modelled as smooth and the wall as rough.



(a) By taking moments around the top of the wall, show that the reaction force at the ground is given by $R_1 = W(1 - \sin\theta)$.

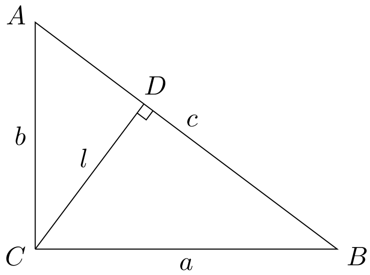
(b) Show that the reaction force at the top of the wall satisfies $R_2 \leq \frac{1}{2}W$.

4123. The curve C has equation $y = xe^x$. A normal N is drawn at the point $x = p$. Show that

(a) if $p = -1$, N does not re-intersect C,

(b) if $p < -1$, N does re-intersect C.

4124. In a right-angled triangle ABC, a perpendicular is drawn from vertex C to side AB.



The *reciprocal Pythagorean theorem* states that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{l^2}.$$

By calculating the area of $\triangle ABC$ in two different ways, or otherwise, prove the theorem.

4125. Prove the following result:

$$\int_1^\infty \frac{\ln x + 1}{x^2} \, dx = 2.$$

You may assume that $\displaystyle\lim_{k\to\infty} \frac{\ln k}{k} = 0$.

4126. A curve, which passes through the origin, has a gradient given by

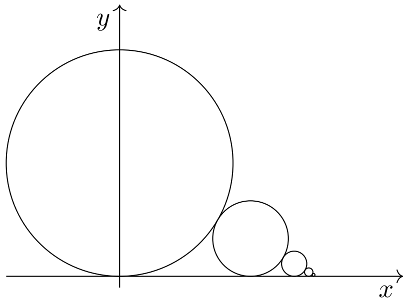
$$\frac{dy}{dx} = \frac{x^2 - 1}{2y}.$$

Show that the curve also passes through $\big(\pm\sqrt{3}, 0\big)$.

4127. Let H represent the number of heads that appear when four fair coins are tossed together. Show that $\mathbb{P}(H \text{ is even}) = \mathbb{P}(H \text{ is odd})$.

4128. An infinite sequence of circles C_n, all tangent to the x axis, have centres (x_n, y_n) and radii r_n. The circles are configured such that:

- Circle C_0 has $x_0 = 0$ and $r_0 = 1$,
- C_{n+1} is tangent to C_n,
- $r_{n+1} = \frac{1}{3} r_n$.



(a) Show that the centres lie on a straight line.

(b) Hence, determine $\lim_{n\to\infty} x_n$.

4129. The function f is defined, for x in radians, as

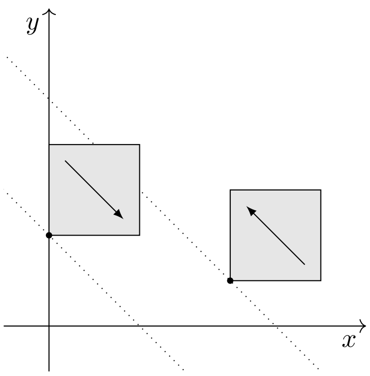
$$\mathrm{f}(x) = x \tan x.$$

By calculating $\mathrm{f}(0), \mathrm{f}'(0)$ and $\mathrm{f}''(0)$, prove that, for small x, $\mathrm{f}(x)$ may be approximated by $\mathrm{g}(x) = x^2$.

4130. Two squares of side length 2 are moving in the (x, y) plane. Their sides are parallel to the axes, and their lower-left vertices have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, where

$$\mathbf{r}_1 = \begin{pmatrix} t \\ 2-t \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} 4-t \\ 1+t \end{pmatrix}.$$

In the diagram, the squares are pictured at $t = 0$, with the paths of their lower-left vertices drawn as dotted lines.



(a) Find the set of t values for which the squares have area in common.

(b) Determine the maximal common area.

4131. Use integration by parts to find

$$\int \sec^2 x \cdot \ln|\sin x| \, dx.$$

4132. A bass piano note has two strings tuned to the same frequency. If the two notes are slightly out of tune, a phenomenon known as a *beat* is heard.

(a) By letting $x = a + b$ and $y = a - b$, prove the *sum-to-product identity*

$$\cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

(b) The out-of-tune notes of the individual strings are modelled by the functions

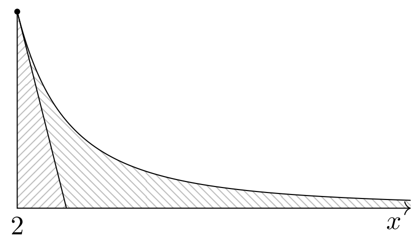
$$t \mapsto \cos pt \text{ and } t \mapsto \cos qt,$$

where p and q differ by only a small amount. By comparing the functions

$$t \mapsto \cos\left(\frac{p+q}{2}t\right) \text{ and } t \mapsto \cos\left(\frac{p-q}{2}t\right)$$

to those of the individual notes, explain why a *beat* is heard as a slow, periodic change in volume.

4133. The graph of $y = (x-1)^{-2}$ is shown, for $x \geq 2$. The straight line is tangent to the curve at $x = 2$.



Show that, over the domain $[2, \infty)$, the tangent line splits the shaded area in the ratio $1 : 3$.

4134. Two sequences U_n and V_n have ordinal formulae, for $n \in \mathbb{N}$, given by

$$U_n = 3n^3 - n^2 - 2n + 1,$$
$$V_n = 2n^3 + n^2,$$

Show that there are exactly two values of n for which $U_n < V_n$.

4135. True or false?

(a) Every cubic has a quadratic factor,

(b) Every quartic has a cubic factor,

(c) Every quintic has a quartic factor.

4136. Show that the following simultaneous equations have exactly one (x, y) solution:

$$\log_{10} x + 2\log_{10} y = 1,$$
$$y = 2x - 3.$$

4137. A graph has equation

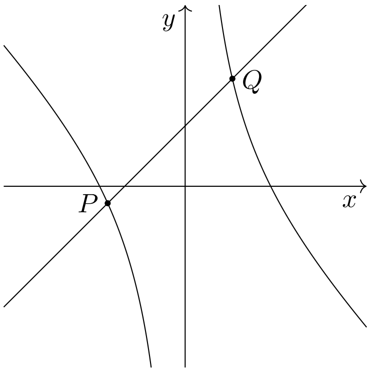
$$y = x^3 e^x.$$

Show that this graph has

(a) a local minimum at $x = -3$,

(b) a point of inflection at $x = 0$.

4138. The curve $xy + x^2 = 2$ is shown, together with a line $y = x + k$. The two intersect at P and Q.

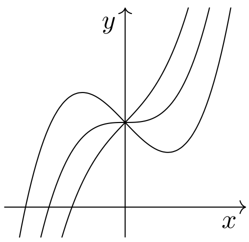


Find all values of k for which $|PQ| = \frac{5\sqrt{2}}{2}$.

4139. A set of n coins is tossed simultaneously. Given that at least $n - 1$ of the coins show tails, find the probability that all n of the coins show tails.

4140. Find the range of $\text{f}(x) = \dfrac{1}{x-1} + \dfrac{1}{(x-1)^2}$.

4141. You are given that $y = x^3 - 3kx + 2$, where k is a constant, intersects the x axis exactly once. Three examples are shown below.



(a) Find the coordinates of any stationary points, in terms of k.

(b) Hence, find the set of possible values of k.

4142. A polynomial function f has, for some $a \neq b$,

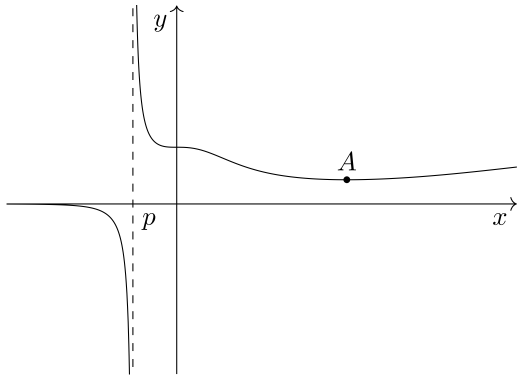
$$\text{f}(a) = \text{f}'(a) = \text{f}(b) = \text{f}'(b) = 0.$$

Prove that f must have degree $n \geq 4$.

4143. Polygon P is a convex n-gon, where $n \geq 5$. The interior angles of P are in arithmetic progression. Show that every interior angle θ satisfies

$$\theta > \frac{n-4}{n}\pi.$$

4144. The graph shown is the curve $y = \dfrac{e^x}{x^3 + e^x}$.



(a) The vertical asymptote has equation $x = p$. Determine the value of p, to 4sf.

(b) Find the exact coordinates of the stationary point marked A.

4145. A definite integral is given as

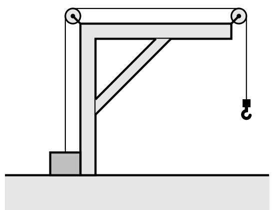
$$\int_0^1 \tan(x^2)\, dx.$$

Show that approximation of this integral by the trapezium rule will always give an overestimate, irrespective of the number of trapezia used.

4146. Show that the curves $y = 2x^2 - 1$ and $x = y^2$ intersect inside the ellipse with equation

$$\left(x + y - \tfrac{4}{5}\right)^2 + 3\left(y - x + \tfrac{1}{2}\right)^2 = 3.$$

4147. A dockyard hoist is modelled as consisting of a fixed arm and a cable which passes over smooth pulleys from a winch engine to a hook. All masses other than those of loads attached to the hook may be assumed to be negligible.



(a) A load of mass 200 kg is attached to the hook.

i. With the load in equilibrium, find the force exerted on each pulley by the cable.

ii. The winch engine now accelerates the load upwards at $0.1g$ ms^{-2}. Find the combined downward force exerted by both pulleys on the hoist arm.

(b) The hoist arm can safely sustain a combined downward force of 20 kN. Find the maximum safe acceleration of an 800 kg load.

4148. A graph has equation $\mathrm{f}_1(x)\,\mathrm{f}_2(y) = 1$, for some functions f_1 and f_2. The graph is rotated 180° around the origin. Write down the equation of the transformed graph.

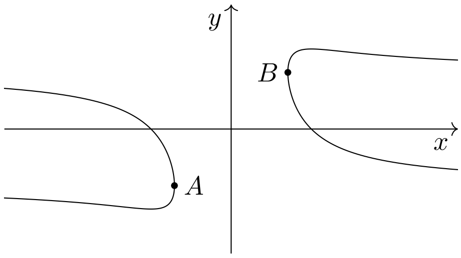
4149. Show there there are no values of a and b which satisfy both the inequality $a^2 + 3b^2 < 10$ and the equation $b = 12 - a^2$.

4150. A quartic curve Q has equation $y = x^4 + x$. A student claims that Q has the origin as a point of inflection.

State, with a reason, whether this is true or not.

4151. A graph is defined parametrically as

$$x = \operatorname{cosec} t, \qquad y = \sin t + \cos t.$$



(a) Show that the Cartesian equation is

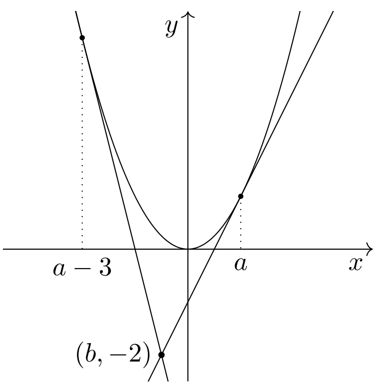
$$(xy - 1)^2 + 1 = x^2.$$

(b) Find the coordinates of the points, labelled A and B, at which the distance from the curve to the y axis is stationary with respect to t.

4152. Find the constant term in the expansion of

$$(x - 1)^4 \left(1 + \tfrac{1}{x}\right)^4.$$

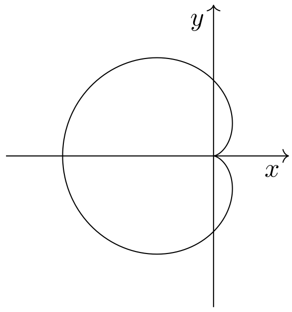
4153. The curve $y = x^2$ has two tangents, at $x = a$ and $x = a - 3$, which meet at the point $(b, -2)$.



Find all possible values of a and b.

4154. Four smooth, uniform spheres of radius r and mass m are hung from the same fixed point, each by a string of length r attached to its surface. The spheres hang in equilibrium, symmetrically. Show that the tension in each string is $T = \sqrt{2}mg$.

4155. The graph shown is a cardioid.



It has parametric definition

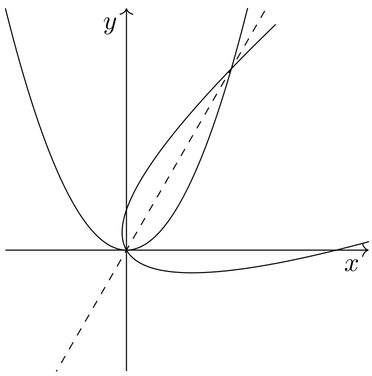
$$x = 2(1 - \cos t)\cos t,$$
$$y = 2(1 - \cos t)\sin t.$$

Verify that its Cartesian equation is

$$\left(x^2 + y^2\right)^2 + 4x\left(x^2 + y^2\right) - 4y^2 = 0.$$

4156. Show that, for all $x \in \mathbb{R}$, $\dfrac{2x^2}{1 + x^2} \leq |x|$.

4157. In the diagram below, the graph $y = x^2$ has been reflected in the line $y = \sqrt{3}x$.



Find the equation of the line of symmetry of the transformed graph.

4158. In a kinematics model, a particle is moving under the action of an attractive force. It has position x at time t. Units are metres and seconds. Initially, the particle is at position $x = 1$. The relationship between velocity and position is modelled as

$$v = x^2.$$

Show that this model breaks down beyond the first second of the motion.

4159. A polynomial $x \mapsto \mathrm{f}(x)$ has $\mathrm{f}'''(x) > 0$ for all $x \in \mathbb{R}$. Show that the range of f is $\mathbb{R}$.

4160. Show that there are no x values satisfying

$$\sin^3 x - 8\cos^2 x + 21\sin x + 26 = 0.$$

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4161. Show that $y = x^3 + y^3$ has points of inflection at $(0, 0)$ and $(0, \pm 1)$.

4162. The independent variables $X_1, ..., X_n$ represent a random sample of size n taken from a population with distribution $X \sim N(\mu, \sigma^2)$. Write down the distributions of

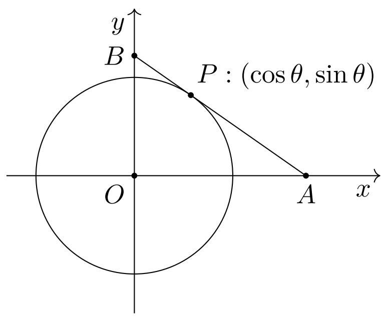
(a) X_1,

(b) $\bar{X}$,

(c) $X_1 + X_2 + ... + X_k$,

(d) $\frac{1}{n}\sum_{i=1}^{n}(aX_i + b)$, for constants a, b.

4163. A unit circle, centred at the origin, has a tangent, which is neither horizontal nor vertical, drawn at point $P : (\cos\theta, \sin\theta)$. The tangent crosses the x axis at A and the y axis at B.



Prove that the area of triangle OAB is $\operatorname{cosec} 2\theta$.

4164. Find $\displaystyle\int \frac{18x^3 - 6x}{3x^3 - 2x^2 - x}\, dx$.

4165. Two functions are defined as $f(x) = 2x + 1$, with domain $\mathbb{R}$, and $g(x) = \sqrt{x}$, with domain $\mathbb{R}^+$. Show that $x = 0$ is the only real root of the equation

$$fg(x) - gf(x) = 0.$$

4166. An exponential model is being used to describe the velocity v of falling objects. At time t, the velocity is given, for positive constants k and $A < B$, by

$$v = A\left(1 - e^{-kt}\right) + Be^{-kt}.$$

(a) Sketch a velocity-time graph.

(b) Interpret the constants A, B and k.

(c) Explain whether this model describes objects falling from rest or not.

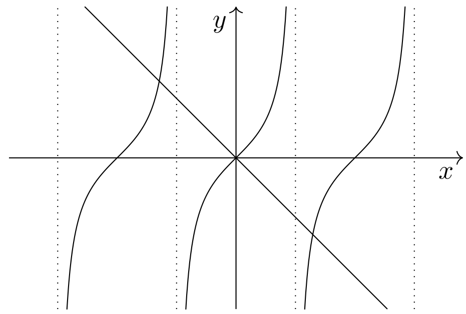
(d) The units of v and A are ms^{-1}. Write down the units of B and k.

(e) Show that the displacement s at time t is

$$s = At + \frac{B - A}{k}\left(1 - e^{-kt}\right).$$

4167. Prove that the function $f(x) = \log_a x - \log_b x$, where $1 < a < b$, is increasing for $x \in (0, \infty)$.

4168. The graph $y = \tan x$ consists of infinitely many periodic sections, separated by asymptotes. Three of these sections are shown below, together with the line $y = -x$.



(a) Show that $y = \tan x$ and $y = -x$ are normal to each other at the origin.

(b) Show that, despite the result of part (a), the line $y = -x$ is not the path of closest approach between the sections of $y = \tan x$ shown.

4169. Determine whether or not the following limits are well defined:

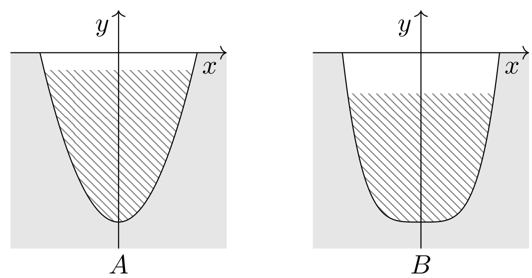
(a) $\displaystyle\lim_{x\to 1} \frac{(x+1)(x-1)}{|x+1|(x-1)}$,

(b) $\displaystyle\lim_{x\to 1} \frac{(x+1)|x-1|}{(x+1)(x-1)}$,

(c) $\displaystyle\lim_{x\to 1} \frac{(x+1)|x-1|}{|x+1|(x-1)}$.

4170. Prove that, if two cubic graphs of the form $y = f(x)$ have four distinct points in common, then they must be the same cubic graph.

4171. A canyon river runs through location A, at which its cross-section is modelled by $y = \frac{1}{2}x^2 - 32$, to location B, at which its cross-section is modelled by $y = \frac{1}{128}x^4 - 32$. Lengths are given in metres, and ground level is modelled as $y = 0$. The cross-sectional area of the water is equal at A and B.



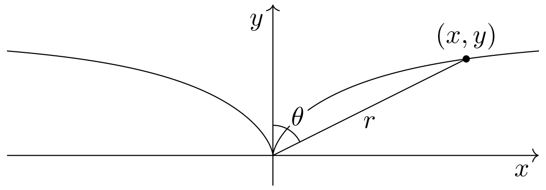
(a) Verify that the two cross-sections have the same width and the same depth.

(b) At A, the surface of the river is 3.27 metres below ground level. Find the equivalent depth of the surface at B.

4172. The sinusoidal function $f(x) = \sin x + \sqrt{3}\cos x$ is not invertible when defined over the domain $\mathbb{R}$. Determine the largest interval $[a, b]$ containing zero over which f is invertible.

4173. Giving your answer in set notation, determine the set of x values for which $e^{3x} - e^{2x} + 5e^x > 0$.

4174. The *cissoid of Diocles* is a classical curve in which distance from the origin r and angle θ (measured, in this question, clockwise from the positive y axis) are related by $r = 2(\sec\theta - \cos\theta)$.



Show that the Cartesian equation of the curve is

$$(x^2 + y^2)y = 2x^2.$$

4175. The height of wheat plants in a field is modelled with a normal distribution $H \sim N(1.1, 0.04)$, with units of metres.

(a) A wheat plant is selected at random. Find the probability it is under a metre tall.

(b) The interval (a, b), which is symmetric around the mean, has a 90% probability of containing any randomly selected plant. Find a and b.

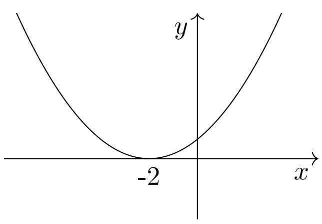
4176. Prove that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$.

4177. Without doing any calculations, sketch the graph $y = (x-a)^\alpha(x-b)^\beta(x-c)^\gamma$, where $0 < a < b < c$, in the cases that

(a) $\alpha = 1$, $\beta = 2$, $\gamma = 3$,

(b) $\alpha = 2$, $\beta = -2$, $\gamma = 2$.

4178. The graph below is of $y = g''(x)$, for some quartic function g defined over $\mathbb{R}$.



(a) Explain why this information is insufficient to locate stationary points.

(b) Write down the set over which g is convex.

(c) State, with a reason, whether or not $x = -2$ is a point of inflection of $y = g(x)$.

4179. Show that five unit squares may be packed without overlap into a square of side length $2 + \sqrt{2}/2$.

4180. Variables x and y are related as

$$x^2y^3 - 2x = 3y.$$

Show that the rate of change of y with respect to x is never zero.

4181. According to the *arithmetic mean-geometric mean* (AM-GM) inequality, for all $a, b \in \mathbb{R}^+$,

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

(a) Show that, if equality holds, then $a = b$.

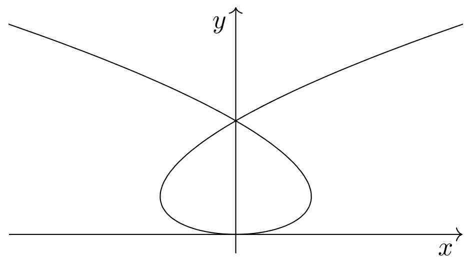
(b) Prove the inequality.

4182. The function $f(x) = \cos x + k\sec x$, where k is a constant, is defined over $[0, \pi/2)$. Show that

(a) when $k = 1$, f is invertible,

(b) when $k = \frac{1}{2}$, f is not invertible.

4183. A parametric curve is given by $x = t^3 - 3t$, $y = t^2$ for $t \in \mathbb{R}$.



The curve has exactly two points A and B with tangents parallel to y. These tangents cross the curve again at P and Q. Show that the area of quadrilateral $ABQP$ is 12.

4184. A stuntman of mass m is hanging, in equilibrium, from a wire. On each side of the stuntman, the wire is inclined at 10° and 15° to the horizontal respectively.

(a) Explain why the contact between the wire and the stuntman cannot be modelled as smooth.

(b) Explain why the reaction force R exerted by the wire on the stuntman is

i. not equal to mg, but

ii. is approximately equal to mg.

(c) Write down the magnitude of the resultant of the two tensions acting on the stuntman.

4185. An equation is given as

$$(x^2-4)^3(x-2) + x(x^2+2x)^3 = 0.$$

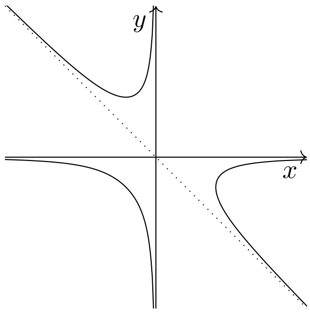
Show that the equation has exactly one real root.

4186. A curve has derivative

$$\frac{dy}{dx} = \frac{e^x + 1}{e^x + 2}.$$

Determine all possible equations of the curve.

4187. A student draws the following graph, claiming it to be the locus of points satisfying $x^2y + xy^2 = 1$.



Show that this claim is incorrect.

4188. A differential equation is given as

$$\frac{d^2y}{dx^2} = 4y.$$

(a) Verify that $y = e^{2x}$ is a solution.

(b) A solution is proposed, in the form $y = \mathrm{f}(x)e^{2x}$ for some function f. Show that f must satisfy

$$\mathrm{f}'(x) + 4\,\mathrm{f}(x) = k.$$

(c) Solve by separation of variables to show that $\mathrm{f}(x) = A + Be^{-4x}$, for constants A, B.

(d) Hence, prove that the general solution of the original differential equation is

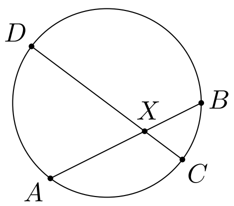
$$y = Ae^{2x} + Be^{-2x}.$$

4189. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

① $\mathrm{f}'(x)$ has a factor of $(x-1)^2$,

② $\mathrm{f}''(x)$ has a factor of $(x-1)$.

4190. The statement of the *intersecting chords theorem* relates to the following diagram:



Prove that $|AX||BX| = |CX||DX|$.

4191. Variables x and $t \geq 0$ are related by

$$t\frac{dx}{dt} = x^2.$$

You are given that $x = 5$ when $t = 1$. Find x as a simplified function of t.

4192. Consider $\mathrm{f}(x) = \arcsin x$, over the usual domain.

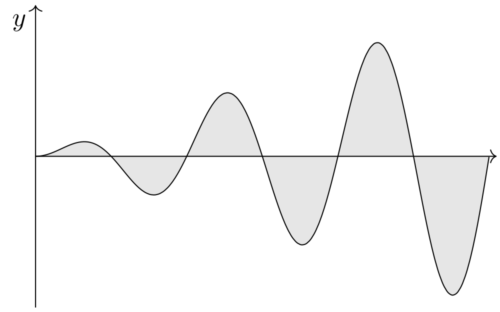
(a) Sketch the graphs $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same set of axes.

(b) Using your graph, show that $\mathrm{f}'(\sin x) = \dfrac{1}{\cos x}$.

(c) Hence, prove that $\mathrm{f}'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

4193. Three dice have been rolled, giving scores X, Y, Z. Show that $\mathbb{P}(X + Z = 2Y \mid X + Y + Z = 12) = \frac{1}{5}$.

4194. The graph $y = x\sin x$, together with the positive x axis, encloses infinitely many regions, as shaded below. These regions have areas $A_1, A_2, ...$.



Show that $A_1, A_2, ...$ is an arithmetic progression.

4195. Prove that a tangent drawn to a cubic at its point of inflection does not intersect the curve again.

4196. Vectors **a** and **b** are defined as

$$\mathbf{a} = \sec\phi\,\mathbf{i} + \tan\phi\,\mathbf{j},$$
$$\mathbf{b} = \tan\phi\,\mathbf{i} + \sec\phi\,\mathbf{j}.$$

(a) Show that

i. $|\mathbf{a}|^2 = |\mathbf{b}|^2 = \dfrac{1+\sin^2\phi}{\cos^2\phi}$,

ii. $|\mathbf{a}-\mathbf{b}|^2 = \dfrac{2(1-\sin\phi)^2}{\cos^2\phi}$.

(b) Hence, show that the angle θ between **a** and **b** satisfies

$$\cos\theta = \frac{\sin\phi}{1+\sin^2\phi}.$$

4197. A function is defined, with θ in radians, as

$$\mathrm{f}(\theta) = 6\sin\theta + 8\cos\theta.$$

A value of θ is chosen at random on the interval $[0, 2\pi)$. Find the following probabilities:

(a) $\mathbb{P}\big(\mathrm{f}(\theta) > 5\big)$,

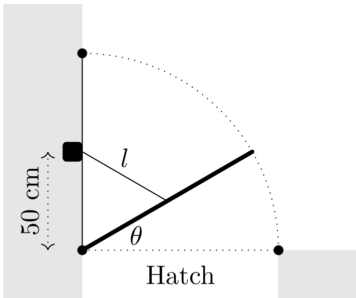
(b) $\mathbb{P}\big(|\mathrm{f}(\theta)| > 5\big)$.

4198. Two quadratic equations are given, with constant coefficients $p, q \in \mathbb{R}$, as

$$x^2 + px + q = 0,$$
$$x^2 - px + q = 0.$$

Show that the sum of the four roots is zero.

4199. A square trapdoor of edge length 1 metre is opened from horizontal to vertical by a mechanism. The mechanism consists of a light cable attached to the midpoint of the trapdoor, which is retracted at constant speed u by a winch. The winch, shown as a black box below, is embedded in a wall, 50 cm above the hinge. In side view, the scenario is:



(a) Show that $\sin\theta = 1 - 2l^2$.

(b) Hence, show that $\dfrac{d\theta}{dt} = \dfrac{2u}{\sqrt{1-l^2}}$.

(c) Determine the time at which the angular speed of opening is greatest.

4200. The end of a student's solution to a differential equation problem is as follows:

$$\int e^y \, dy = \int 2x + 1 \, dy$$
$$\implies e^y = x^2 + x$$
$$\implies y = \ln(x^2 + x) + c.$$

Explain the error and correct it.

——— END OF 42ND HUNDRED ———